Almost sure consensus for multi-agent systems with two level switching structure

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Abstract

In most literatures on the consensus of multi-agent system (MAS), the agents considered are time-invariant. However in many cases (e.g. aerocrafts), the dynamics of agents have the characteristic of switching. Moreover, such switching in agent might be accompanied with the change of the interconnection topology of this MAS. This paper proposes a new model of two level switching structure to describe this type of MAS. The switching in the low level, which represents the variation of the agent dynamics, is deterministic and controllable. The switching in the upper level, which reflects the random change of the topology, fits for a Markov chain. Besides, the transition probability of the Markov chain in the upper level varies accordingly with the low level switching. This paper deals with the almost sure (AS) consensus for the MAS with two level switching. By analyzing the transient features of discrete-time Markov chain and based on the method of dwell time, a sufficient condition of AS consensus is proposed.

Keywords: multi-agent systems; consensus; agent switching; topology switching; dwell time

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1. Introduction

Recent years have witnessed the increasing attentions on the studies of multi-agent system (MAS) which have broad applications including cooperative control of unmanned vehicles [1-2], formation control of mobile robots [3-4], flocking [5], synchronization [6] and consensus [7-9]. A consensus problem is to design distributed protocols for the agents to reach an agreement of state.

Notice the fact that the communication link are often unstable due to external noise, and random fault of components etc., in recent studies the topology of MAS are often assumed to have the feature of stochastic switching. These stochastic switchings are often modeled by Bernoulli process [10] or Markov processes [11], then the problem of stochastic consensus is investigated. Stochastic consensus includes the following two classes: mean square (MS) and almost sure (AS). MS consensus requires the expectation of the state difference between any two agents converge to zero asymptotically. AS consensus only needs the state trajectories of each agents converge to a common point with probability one, which is generally more practical than MS consensus. For the case that part of the connection topologies are disconnected, sufficient conditions for AS consensus and MS consensus of linear MAS are given based on the dwell time and the average dwell time methods [12]. Furthermore, even if each possible topology of MAS is disconnected, MS consensus [13-15] and AS consensus [16-17] can still be achieved as long as the union of topologies are joint connected.

Most of the literatures on MAS are aiming at time-invariant agents. However, in practice the agents often appear the characteristics of time-variant or stochastic. For example, the dynamics of an aerocraft can change significantly during flight, thus it is reasonable to model this aerocraft system by switched systems [18]. The reasons for causing the dynamics change come from multiple aspects, e.g., separation of boosters from the rockets, discard of auxiliary fuel tanks, and sudden change of flight attitude (i.e. pitch, roll and yaw). Particularly, some factors, such as the drastic variation of attitude, can not only change the dynamics of aerocraft, but also affect the quality of the communications with other agents and thereby influence the interconnection topology[19].

This paper investigates the AS consensus problem for MAS by taking the agent switching and the topology switching into account simultaneously.

A model with the structure of two level switching is developed to describe such MAS. The low level switching, i.e. the switching of agent dynamics, is supposed to be deterministic and synchronous. The stochastic topology switching in the upper level is assumed to fit for a discrete-time Markov chain. Furthermore, the transition probability of the Markov chain varies along with the low level switching. The architecture of this MAS is shown in Figure.1. As an example, this MAS is composed of four agents, and the dynamics of each agent has two patterns. The switching rule of agent dynamics, i.e the lower level switching, is denoted by $\gamma(k)$. Besides, the interconnection topologies of the MAS have three possible modes, the switching among which is conducted by a Markov chain $\{\sigma(k)\}$ with the transition probability $P^{[\gamma(k)]}$. This makes the upper level switching.

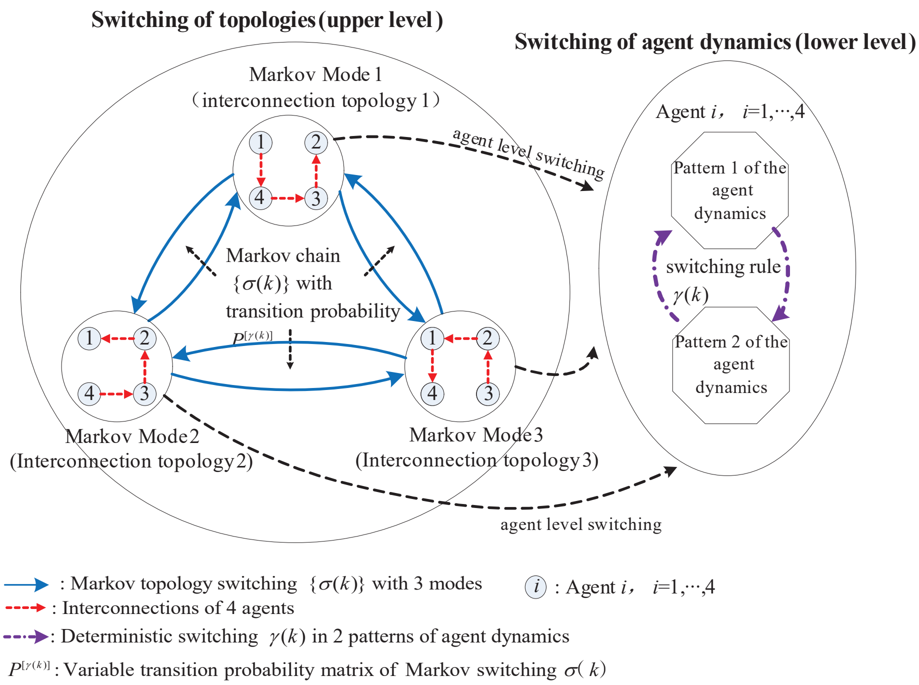

Figure 1: MAS with the structure of two level switching

It should be noted that similar stochastic systems with two level switching are also developed recently [20-21]. A Markovain jump linear system with overlapping group modes is proposed [20], and then the stochastic stability as well as H-infinity control are investigated. In architecture, different from this paper, the switchings in the two levels in [20] are both Markovain, and meanwhile there exists special overlapping local modes [20]. Another two level switching system called switching-Markov jump system(S-MJS) is proposed in [21]. In S-MJS, since the switching in the upper level is not affected by the low level switching, therefore the ergodicity of the Markov chain is kept. By applying this ergodicity, sufficient conditions of almost sure stability are derived for S-MJS [21]. However, in this paper, due to the transition probability of Markovain switching is changeable, it breaks the ergodicity of Markov chain. That is, the results in [21] cannot be utilized to tackle the more general case of this paper. By analyzing the transient features of a discrete-time Markov chain, this paper solves this problem and the ergodicity is not needed any more in the derivation of the main result. A sufficient condition of the AS consensus of MAS is then proposed based on the method of dwell time.

The rest of this paper is organized as follows. Section 2 provides a brief introduction of graph theory. Section 3 formulates the problem. In Section 4, a lemma on the transient characteristics of a Markov process is given firstly. Then a sufficient condition for MAS to reach AS consensus is presented. Section 5 provides an example to demonstrate the effectiveness of the proposed results. Section 6 concludes the paper. Appendix 1 presents the design steps of the control parameters in consensus protocol. Appendix 2 gives the proof of Lemma 2.

Notations:

$R^{n\times n}$	Set of $n \times n$ real matrices
R^n	Set of n-dimension real column vectors
I_n	n-dimension identity matrix
$\|A\|$or $\|x\|$	Spectral norm of matrix A (or 2-norm of vector x)
A^T (or x^T)	Transpose of matrix A (or vector x)
$\rho(A)$	Spectral radius of a square matrix A
$A > (\geq)B$	$A - B$ is positive (semi-)definite
$\otimes$	Kronecker product, which satisfies: $(A \otimes B)(C \otimes D) = AC \otimes BD$; $(A \otimes B)^T = A^T \otimes B^T$

2. Graph theory

In this section, some basic concepts on graph theory are introduced.

The topology of MAS is described by a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ is a finite nonempty set of agents, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges, $\mathcal{A} = [a_{ij}] \in R^{N \times N}$ is the weighted adjacency matrix satisfying that $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. An edge (j, i) in $\mathcal{E}$ means that agent i can receive information from agent j. Here, we exclude the self-connection, i.e. $(i, i) \notin \mathcal{E}$ and $a_{ii} = 0$. A sequence of edges (i_1, i_2), (i_2, i_3),. . ., (i_{k-1}, i_k) with $(i_{j-1}, i_j) \in \mathcal{E}$, $\forall j \in \{2, \ldots, k\}$ is called a directed path from agent i_1 to agent i_k. A digraph $\mathcal{G}$ is called connected (or contain a directed spanning tree) if there is a node $i \in \mathcal{V}$ such that there exists a directed path from node i to all other nodes in $\mathcal{G}$, otherwise, $\mathcal{G}$ is said to be disconnected. The Laplacian matrix $L = [l_{ij}] \in R^{N \times N}$ of digraph $\mathcal{G}$ is defined as $l_{ij} = -a_{ij}$, $l_{ii} = \sum_{j=1, j\neq i}^{j=N} a_{ij}$. A lemma on the properties of Laplacian matrix is given as below.

Lemma 1[22]: All the eigenvalues of Laplacian matrix L have nonnegative real parts. Zero is an eigenvalue of L with $\mathbf{1}$ as the right eigenvector, thus referred as $\lambda_{\mathbf{1}}(L) = 0$. Furthermore, the algebraic multiplicity of the zero eigenvalue $\lambda_{\mathbf{1}}(L)$ is 1 if and only if the corresponding digraph $\mathcal{G}$ is connected.

3. Problem formulation

Consider the following MAS with N agents. Agent i, $i \in \mathcal{N} = \{1, \ldots, N\}$ is described as

$$x_i(k+1) = A^{[\gamma(k)]} x_i(k) + B^{[\gamma(k)]} u_i(k), \tag{1}$$

where $x_i(k) \in R^n$ and $u_i(k) \in R^m$. The dynamic of the agent has s pattern, denoted by $\left(A^{[1]}, B^{[1]}\right), \left(A^{[2]}, B^{[2]}\right), \cdots, \left(A^{[S]}, B^{[S]}\right)$ respectively. The pair $(A^{[j]}, B^{[j]})$ is stabilizable, $\forall j \in \mathcal{S}$, $\mathcal{S} := \{1, \ldots, S\}$. Deterministic function $\gamma(k) \in \mathcal{S}$ governs the switching of agent dynamics. In this paper, $\gamma(k)$ is assumed to be synchronous for all the agents.

Instant k is called a deterministic switching instant (DSI), if $\gamma(k-1) \neq \gamma(k)$. Denote $D(k', k'')$ as the number of DSI which take place in the interval $[k', k'')$. $T_q^{[j]}$ is the q-th successive sojourn time of the j-th agent pattern,

$T_{D\min}^{[j]}$ is the minimal dwell time of the j-th agent pattern. Clearly, $T_q^{[j]} \geq T_{D\min}^{[j]}, q = 1, 2, \cdots$.

The Markovain switching of topology is described by the digraph $\mathcal{G}_{\sigma(k)} = (\mathcal{V}, \mathcal{E}_{\sigma(k)}, \mathcal{A}_{\sigma(k)}) \in \mathcal{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_q\}$, in which the connected graphs are denoted by $\mathcal{G}_1, \ldots, \mathcal{G}_r$. The transition probability matrix of Markov chain $\{\sigma(k)\}$ is not constant but varies with deterministic switching $\gamma(k)$, thus it is denoted by $P^{[\gamma(k)]}$. In this paper, $P^{[j]}$ is irreducible and aperiodic, $\forall j \in \mathcal{S}$. Therefore it has unique invariant distributions $\boldsymbol{\pi}^{[j]} = \begin{bmatrix} \pi_1{}^{[j]} & \pi_2{}^{[j]} & \cdots & \pi_q^{[j]} \end{bmatrix}^T$ which can be obtained by

$$\begin{cases} \boldsymbol{\pi}^{[j]T} = \boldsymbol{\pi}^{[j]T} P^{[j]}, \\ \sum_{i=1}^{q} \pi_i^{[j]} = 1. \end{cases} \tag{2}$$

Definition 1: MAS (1) with deterministic agent switching $\gamma(k)$ and Markovain topology switching $\sigma(k)$ is said to reach an almost sure consensus if

$$\Pr\left(\lim_{k\to\infty} \|x_i(k) - x_j(k)\| = 0\right) = 1, \ \ i, j \in \mathcal{N}$$

holds for any initial condition(i.e. initial state $x_i(0)$, initial deterministic switching position $\gamma(0)$, the initial distribution of $\boldsymbol{f}^{[\gamma(0)]}$).

Definition 2: A matrix F is called Schur if the spectral radius $\rho(F) < 1$.

Construct the piecewise consensus protocol for the i-th agent:

$$u_i(k) = cK^{[\gamma(k)]} \sum_{j\in\mathcal{N}} a_{\sigma(k)}^{ij} \left(x_j(k) - x_i(k)\right), \tag{3}$$

where $c \in R^+$ is the coupling strength, $K^{[\gamma(k)]} \in R^{m\times n}$ is the controller gain matrix. Parameter $a_{\sigma(k)}^{ij}$ is the (i, j)-th element of adjacency matrix $\mathcal{A}_{\sigma(k)}$ of $\mathcal{G}_{\sigma(k)}$.

Let $X = [x_1^T, \ldots, x_N^T]^T$. Substitute consensus protocol (3) into (1),

$$X(k+1) = (I_N \otimes A^{[\gamma(k)]} - cL_{\sigma(k)} \otimes B^{[\gamma(k)]} K^{[\gamma(k)]})X(k), \tag{4}$$

where $L_{\sigma(k)}$ is the Laplacian matrix of digraph $\mathcal{G}_{\sigma(k)}$.

Introduce the following variable transformation [23]

$$\xi(k) = (T \otimes I_n)\, X(k),$$

where

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}.$$

Clearly, $T^{-1} = T$. Rewrite (4) with respect to ξ,

$$\xi(k+1) = \left(I_N \otimes A^{[\gamma(t)]} - cTL_{\sigma(k)}T \otimes B^{[\gamma(t)]}K^{[\gamma(t)]}\right)\xi(k), \tag{5}$$

Denote $\xi = \begin{bmatrix} \xi_1 \\ \xi_R \end{bmatrix}$, $\xi_1 := x_1$ and $\xi_R := \left[(x_1 - x_2)^T, (x_1 - x_3)^T, \cdots, (x_1 - x_N)^T\right]^T$. Let $W = \begin{bmatrix} \mathbf{1}_{N-1} & -I_{N-1} \end{bmatrix}$ and $Y = \begin{bmatrix} \mathbf{0}_{N-1} & -I_{N-1} \end{bmatrix}^T$. Denote $l^1_{\sigma(k)}$ as the first row of $L_{\sigma(k)}$. Noticing $L_{\sigma(k)}\mathbf{1}_N = 0$, then (5) can be decomposed as

$$\xi_1(k+1) = A^{[\gamma(k)]}\xi_1(k) - cl^1_{\sigma(k)}YB^{[\gamma(k)]}K^{[\gamma(k)]}\xi_R(k), \tag{6}$$

$$\begin{aligned} \xi_R(k+1) &= \left(I_{N-1} \otimes A^{[\gamma(k)]} - cWL_{\sigma(k)}Y \otimes B^{[\gamma(k)]}K^{[\gamma(k)]}\right)\xi_R(k) \\ &:= F^{[\gamma(k)]}_{\sigma(k)}\xi_R(k). \end{aligned} \tag{7}$$

Thus by the virtue of the transformation (5), the problem of AS consensus of MAS (1) is transformed to check the AS stability of (7).

The steps of computing the control parameters c and $K^{[\gamma(k)]}$ in protocol (3) are given in Appendix 1. Moreover, by Lemma 3 in Appendix 1, it is clear that with the designed c and $K^{[j]}$, matrix $F^{[j]}_i$ is Schur if $\mathcal{G}_i$ is connected. The convergence rates of $F^{[j]}_i$ can be estimated as below.

Lemma 1: For a linear time-invariant system $\xi(k+1) = F\xi(k)$, $\psi(k', k)$ is the transition function, then there exist scalar parameters α, β such that

$$\ln \|\psi(k', k)\| \le \alpha + \beta\,(k - k') \tag{8}$$

α and β can be obtained as follows:

Find matrix $Q > 0$,$M > 0$, scalar $\nu > 0$ to satisfy

$$F^T Q F - \nu^2 Q + M = 0, \tag{9}$$

then,

$$\alpha = \frac{1}{2} \ln \left[\lambda_{\max}(Q) / \lambda_{\min}(Q) \right], \beta = \ln \nu \tag{10}$$

Proof: Notice that

$$\begin{aligned}
\lambda_{\min}(Q)\, \xi^T(k)\xi(k) \le \xi^T(k) Q \xi(k) &= \xi^T(k-1) F^T Q F \xi(k-1) \\
&< \nu^2 \xi^T(k-1) Q \xi(k-1) \\
&\le \cdots < \nu^{2(k-k')} \xi^T(k') Q \xi(k') \\
&\le \nu^{2(k-k')} \lambda_{\max}(Q)\, \xi^T(k')\xi(k')
\end{aligned}$$

Thus for $\forall \xi(k') \neq 0$,

$$\frac{\|\xi(k)\|}{\|\xi(k')\|} < \sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}} \nu^{k-k'}$$

Noticing $\xi(k) = \psi(k,k')\xi(k')$, it follows

$$\|\psi(k,k')\| = \max_{\xi(k') \neq 0} \frac{\|\psi(k,k')\xi(k')\|}{\|\xi(k')\|} = \max_{\xi(k') \neq 0} \frac{\|\xi(k)\|}{\|\xi(k')\|} < \sqrt{\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}} \nu^{k-k'}$$

This completes the proof. □

4. Main results

In this section, a lemma on the stochastic characteristics of Markov chain is presented firstly. Then based on the method of dwell time and by using the preliminary lemmas, sufficient condition of AS consensus for MAS (1) are proposed.

The following Lemma presents the transient characteristic of a discrete-time Markov chain.

Lemma 2: Consider an irreducible and aperiodic Markov chain $\sigma(k)$ with S modes. $P = [p_{ij}]_{S \times S}$ is the transition probability. Denote $\pi =$

$\begin{bmatrix} \pi_1 & \cdots & \pi_S \end{bmatrix}^T$ the unique invariant distribution. $E_{\boldsymbol{f}}[\cdot]$ is the expectation operator with respect to the initial probability distribution $\boldsymbol{f} = [f_1 \cdots f_S]^T$, where $f_i = \Pr\{\sigma(0) = i\}$, $1 \leq i \leq S$. Denote $T_i(0,k)$ the accumulated sojourn time of mode i in interval $[0,k)$. $N_i(0,k)$ is the number of the activations of mode i in $[0,k)$. Then, the following are satisfied for each mode i,

$$E_{\boldsymbol{f}}[T_i(0,k)] = f_i + \pi_i(k-1) + \boldsymbol{f}^T(I - P^k)\left[(I - P + \mathbf{1}_{S\times S})^{-1}\right]_i \tag{11}$$

$$\begin{aligned} &E_{\boldsymbol{f}}[N_i(0,k)] \\ &\quad = (1-p_{ii})\left(f_i + \pi_i(k-1) + \boldsymbol{f}^T(I-P^k)\left[(I-P+\mathbf{1}_{S\times S})^{-1}\right]_i\right) \\ &\qquad + p_{ii}\boldsymbol{f}^T\left[P^{k-1}\right]_i \end{aligned} \tag{12}$$

where $[*]_i$ represents the i-th column of the matrix $*$.

Proof: See Appendix 2.

Remark 1: The bounds of $E_{\boldsymbol{f}}[T_i(0,k)]$ and $E_{\boldsymbol{f}}[N_i(0,k)]$ can be estimated as below:

For arbitrary $\boldsymbol{f}$ and k, is clear that $0 < \boldsymbol{f}^T\left[P^{k-1}\right]_i \leq 1$. Denote $[c_1 \ c_2 \ \cdots \ c_S] := f^T(I-P^k)$, $[b_{1i} \ b_{2i} \ \cdots \ b_{Si}]^T := \left[(I-P+1_{S\times S})^{-1}\right]_i$, where $\sum_{i=1}^S c_i = 0$, $-1 \leq c_i \leq 1$, and

$$\boldsymbol{f}^T(I-P^k)\left[(I-P+\mathbf{1}_{S\times S})^{-1}\right]_i = \sum_{j=1}^S c_j b_{ji}$$

Notice transition probability P is given,hence b_{ji} is known. Rearrange $b_{1i}, b_{2i}, \ldots, b_{Si}$ into $\tilde{b}_{1i} \geq \tilde{b}_{2i} \geq \ldots \geq \tilde{b}_{Si}$, then,

$$\begin{aligned} \delta_i := \max_{c_1,c_2,\cdots,c_S} \sum_{j=1}^S c_j b_{ji} &= -\min_{c_1,c_2,\cdots,c_S} \sum_{j=1}^N c_j b_{ji} \\ &= \begin{cases} \sum_{j=1}^{S/2} \tilde{b}_{ji} - \sum_{j=1+S/2}^{S} \tilde{b}_{ji}, & \text{If } S \text{ is even number} \\ \sum_{j=1}^{(S-1)/2} \tilde{b}_{ji} - \sum_{j=2+(S-1)/2}^{S} \tilde{b}_{ji}, & \text{If } S \text{ is odd number} \end{cases} \end{aligned} \tag{13}$$

This leads to,

$$f_i + \pi_i(k-1) - \delta_i \leq E_{\boldsymbol{f}}\left[T_i\left(0,k\right)\right] \leq f_i + \pi_i(k-1) + \delta_i \tag{14}$$

$$E_{\boldsymbol{f}}\left[N_i\left(0,k\right)\right] \leq \left(1-p_{ii}\right)\left(f_i + \pi_i(k-1) + \delta_i\right) + p_{ii} \tag{15}$$

□

Then a result on the sufficient condition of AS consensus for MAS (1) is presented.

Theorem 1: Consider MAS (1) with deterministic switching $\gamma(k)$ of agent dynamics and Markovian topology jumping $\sigma(k)$, $\gamma(k) \in \{1,2,\cdots,S\}$, $\sigma(k) \in \{1,2,\cdots,q\}$. MAS (1) can reach almost sure consensus under the protocol (3), if the following inequalities are satisfied for $\forall j$, $1 \leq j \leq S$,

$$\varsigma^{[j]} < 0, \ , T_{D\min}^{[j]} + \frac{\mu^{[j]}}{\varsigma^{[j]}} > 0 \tag{16}$$

where the minimal dwell time $T_{D\min}^{[j]}$ and the transition probability $p_{ii}^{[j]}$ are defined in Section 3,

$\varsigma^{[j]} = \sum_{i=1}^{q}\left[\alpha_i^{[j]}\pi_i^{[j]}\left(1-p_{ii}^{[j]}\right) + \beta_i^{[j]}\pi_i^{[j]}\right]$,

$\mu^{[j]} = \lambda_{\max}^{[j]} + \eta^{[j]}$,

$\lambda_{\max}^{[j]} = \max\{\alpha_i^{[j]} - \alpha_i^{[j]}p_{ii}^{[j]} + \beta_i^{[j]}, 1 \leq i \leq q\}$,

$\eta^{[j]} = -\varsigma^{[j]} + \sum_{i=1}^{q}\left[\alpha_i^{[j]}\left(1-p_{ii}^{[j]}\right)\delta_i^{[j]} + \alpha_i^{[j]}p_{ii}^{[j]}\right] - \sum_{i=1}^{r}\beta_i^{[j]}\delta_i^{[j]} + \sum_{i=r+1}^{q}\beta_i^{[j]}\delta_i^{[j]}$
$\varsigma^{[j]} = \sum_{i=1}^{q}\left[\alpha_i^{[j]}\pi_i^{[j]}\left(1-p_{ii}^{[j]}\right) + \beta_i^{[j]}\pi_i^{[j]}\right]$,

the parameters $\alpha_i^{[j]}$, $\beta_i^{[j]}$ and $\delta_i^{[j]}$ are given by Lemma 1 and (13) of Remark 1 respectively.

Proof: Assume the agent dynamics is in pattern j (i.e. $\gamma(k) = j$) during $k \in [k', k'')$. Denote $N_i^{[j]}\left(k', k''\right)$ and $T_i^{[j]}\left(k', k''\right)$ the number of the visits and the total sojourn time of i-th Markov mode since $\gamma(k) = j$ in $[k', k'')$.

By Lemma 1, one can obtain that

$$\ln \|\psi (k', k'')\| \le \sum_{i=1}^{q} \left[\alpha_i^{[j]} N_i^{[j]} (k', k'') + \beta_i^{[j]} T_i^{[j]} (k', k'')\right] \tag{17}$$

Next, for arbitrary probability distribution $\boldsymbol{f}^{[j]} = \left[\begin{array}{ccc} f_1^{[j]} & \cdots & f_q^{[j]} \end{array}\right]^T$, $f_i^{[j]} = \Pr\{\sigma(k') = i\}$, $1 \le i \le q$, the boundaries of $N_i^{[j]} (k', k'')$ and $T_i^{[j]} (k', k'')$ are given by (14)(15) as follows,

$$f_i^{[j]} + \pi_i^{[j]}(\Delta k - 1) - \delta_i^{[j]} \le E_{\boldsymbol{f}^{[j]}} \left[T_i^{[j]} (k', k'')\right] \le f_i^{[j]} + \pi_i^{[j]}(\Delta k - 1) + \delta_i^{[j]}$$

$$E_{\boldsymbol{f}^{[j]}} \left[N_i^{[j]} (k', k'')\right] \le \left(1 - p_{ii}^{[j]}\right) \left(f_i^{[j]} + \pi_i^{[j]}(\Delta k - 1) + \delta_i^{[j]}\right) + p_{ii}^{[j]}$$

where $\Delta k = k'' - k'$, $P^{[j]}$ is the matrix of transition probability, $\boldsymbol{\pi}^{[j]}$ is the unique invariant distribution.

Thus, it follows from (17) that

$$E_{\boldsymbol{f}^{[j]}} (\ln \|\psi (k', k'')\|) \le \sum_{i=1}^{q} \lambda_i^{[j]} f_i^{[j]} + \Delta k \varsigma^{[j]} + \eta^{[j]} \tag{18}$$

where $\lambda_i^{[j]} := \alpha_i^{[j]} - \alpha_i^{[j]} p_{ii}^{[j]} + \beta_i^{[j]}$, $\varsigma^{[j]} := \sum_{i=1}^{q} \left[\alpha_i^{[j]} \pi_i^{[j]} \left(1 - p_{ii}^{[j]}\right) + \beta_i^{[j]} \pi_i^{[j]}\right]$, $\eta^{[j]} := -\varsigma^{[j]} + \sum_{i=1}^{q} \left[\alpha_i^{[j]} \left(1 - p_{ii}^{[j]}\right) \delta_i^{[j]} + \alpha_i^{[j]} p_{ii}^{[j]}\right] - \sum_{i=1}^{r} \beta_i^{[j]} \delta_i^{[j]} + \sum_{i=r+1}^{q} \beta_i^{[j]} \delta_i^{[j]}$.

Let $\lambda_{\max}^{[j]} := \max\left\{\lambda_i^{[j]}, 1 \le i \le q\right\}$. Notice $\sum_{i=1}^{q} f_i^{[j]} = 1$, then,

$$E_{\boldsymbol{f}^{[j]}} (\ln \|\psi (k', k'')\|) \le \lambda_{\max}^{[j]} + \Delta k \varsigma^{[j]} + \eta^{[j]} \tag{19}$$

Assume that there are h deterministic switchings in the period $[0, k)$. The sequence of these deterministic switching is represented by $W = (d_0, \gamma_0)$, $(d_1, \gamma_1), \cdots, (d_h, \gamma_h)$, where $d_0 = 0$ and d_i is the deterministic switch-in (DSI) instant, $\gamma(d_i) = \gamma_i$. Due to the agent dynamics keep unchanged in each time interval $[d_0, d_1), [d_1, d_2), \ldots, [d_h, k)$, that is $\gamma(k)$ is fixed in these intervals. It follows from (19) that

$$\begin{aligned}E_{\boldsymbol{f}^{[\gamma_0]}}\left(\ln\|\psi(0,k)\|\right) &\le E_{\boldsymbol{f}^{[\gamma_h]}}\left(\ln\left\|\psi\left(d_h,k\right)\right\|\right)\\ &\quad+\sum_{m=0}^{h-1}E_{\boldsymbol{f}^{[\gamma_m]}}\left(\ln\|\psi(d_m,d_{m+1})\|\right)\\ &\le\sum_{j=1}^{S}\left[\varsigma^{[j]}T_D^{[j]}(0,k)+\mu^{[j]}D^{[j]}(0,k)\right]\end{aligned}\tag{20}$$

where $\boldsymbol{f}^{[\gamma_q]}$ is the absolute probability distribution at instant d_q, $0\le q\le h$, $\mu^{[j]}=\lambda_{\max}^{[j]}+\eta^{[j]}$, $T_D^{[j]}(0,k)$ and $D^{[j]}(0,k)$ are the total dwell time and the total visits of j-th agent pattern in the interval $[0,k)$, respectively.

Let $r^{[j]}(k)$ be the fraction of total sojourn time of j-th agent pattern in the interval $[0,k)$. Then it holds that

$$T_D^{[j]}(0,k)=r^{[j]}(k)k \tag{21}$$

On the other hand, the definition of minimal dwell time implies that

$$D^{[j]}(0,k)\le r^{[j]}(k)k\Big/T_{D\min}^{[j]} \tag{22}$$

Noticing $\mu^{[j]}>0$ and by substituting (21) (22) into (20), it follows

$$E_{\boldsymbol{f}^{[\gamma_0]}}\left(\ln\|\psi(0,k)\|\right)\le k\left[\sum_{j=1}^{S}r^{[j]}(k)\left(\varsigma^{[j]}+\frac{\mu^{[j]}}{T_{D\min}^{[j]}}\right)\right] \tag{23}$$

From condition (17), one can see that

$$\begin{aligned}E_{\boldsymbol{f}^{[\gamma_0]}}\left(\lim_{k\to\infty}\frac{\ln\|\xi_R(k)\|}{k}\right)&\le E_{\boldsymbol{f}^{[\gamma_0]}}\left(\lim_{k\to\infty}\frac{\ln\|\psi(0,k)\|}{k}\right)\\ &\le\sum_{j=1}^{s}\bar{r}^{[j]}\left(\varsigma^{[j]}+\frac{\mu^{[j]}}{T_{D\min}^{[j]}}\right)<0\end{aligned}\tag{24}$$

where $\bar{r}^{[j]}=\lim_{k\to\infty}r^{[j]}(k)\ge 0$.

This leads to

$$\Pr\left(\lim_{k\to\infty}\|\xi_R(k)\|=0\right)=1,$$

which guarantees the AS stability [23][24] of (7). Thus the AS consensus of MAS (1) are be reached. This completes the proof. □

Remark 2: Theorem 1 gives a sufficient condition to check the AS consensus of MAS(1) which contains both deterministic agent switching and Markov topology jumping. Note that the transition probability varies with the deterministic switching, therefore the ergodic law of large numbers cannot be used in the case. This issue is solved by using the derived $E_{\boldsymbol{f}}\left[T_i\left(0,k\right)\right]$ and $E_{\boldsymbol{f}}\left[N_i\left(0,k\right)\right]$ given in Lemma 2.

5. Numerical examples

Consider MAS (1) with four agents. The dynamics of each agent has two patterns:

$$A^{[1]}=\begin{bmatrix}1&0&0\\-1&0.1&0.9\\1.4&0&-0.06\end{bmatrix},\ B^{[1]}=\begin{bmatrix}1\\-0.5\\0.4\end{bmatrix}$$

$$A^{[2]}=\begin{bmatrix}1&0.1&0\\-1&0.1&0.7\\1.6&0&0\end{bmatrix},\ B^{[2]}=\begin{bmatrix}0.9\\-0.6\\0.3\end{bmatrix}$$

The stochastic switching of interconnection topology is described by a Markov chain with two modes $\mathcal{G}_1,\ \mathcal{G}_2$, as shown in Figure 2. The weight on each edge is 1. Transition probability $P^{[\gamma(k)]}$ is

$$P^{[1]}=\begin{bmatrix}0.9&0.1\\0.7&0.3\end{bmatrix},\ P^{[2]}=\begin{bmatrix}0.9&0.1\\0.8&0.2\end{bmatrix}$$

The invariant distributions are

$$\boldsymbol{\pi}^{[1]}=\begin{bmatrix}\frac{7}{8}&\frac{1}{8}\end{bmatrix}^T,\ \boldsymbol{\pi}^{[2]}=\begin{bmatrix}\frac{8}{9}&\frac{1}{9}\end{bmatrix}^T$$

Clearly, both $A^{[1]}$ and $A^{[2]}$ are not Schur. The steps of the design of consensus protocol as well as the rule of deterministic switching are presented as follows:

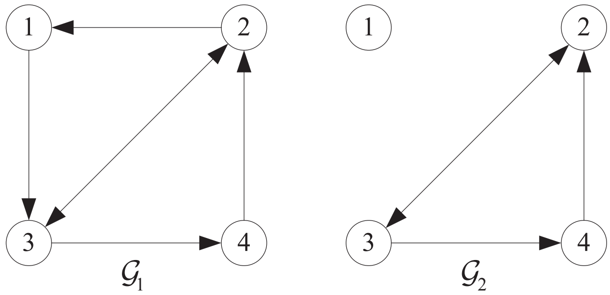

Figure 2: The graphs of the interconnection topologies

1) Construct the corresponding auxiliary equation (7) from combining the agent dynamics (1), consensus protocol (3) and Laplacian matrix $L_{\sigma(k)}$, where $\sigma(k)= 1, 2$, $\gamma(k)= 1, 2$.

2) Solve the Riccati equality (25) in Appendix 1, and obtain

$$Q^{[1]} = \begin{bmatrix} 2.96 & 0.07 & -0.59 \\ 0.07 & 0.81 & 0.06 \\ -0.59 & 0.06 & 0.83 \end{bmatrix},\ Q^{[2]} = \begin{bmatrix} 2.93 & 0.05 & -0.35 \\ 0.05 & 0.82 & 0.07 \\ -0.35 & 0.07 & 0.57 \end{bmatrix}$$

Choose $c = 0.5$ by (26)in Appendix 1.

3) By (27), the distributed feedback laws $K^{[\gamma(k)]}$ are

$$K^{[1]} = \begin{bmatrix} 0.95 & -0.01 & -0.10 \end{bmatrix},\ K^{[2]} = \begin{bmatrix} 1.07 & 0.08 & -0.12 \end{bmatrix}$$

4) It can seen that $F_1^{[1]}$, $F_1^{[2]}$ are Schur while $F_2^{[1]}$ and $F_2^{[2]}$ are not. The parameters $\alpha_{\sigma(k)}^{[\gamma(k)]}$ and $\alpha_{\sigma(k)}^{[\gamma(k)]}$ can be computed by Lemma 1,

$$\begin{cases} \alpha_1^{[1]}=2.01 \\ \beta_1^{[1]}= -0.69 \end{cases},\ \begin{cases} \alpha_2^{[1]} = 0.57 \\ \beta_2^{[1]} = 0.18 \end{cases},\ \begin{cases} \alpha_1^{[2]} = 1.75 \\ \beta_1^{[2]} = -0.60 \end{cases},\ \begin{cases} \alpha_2^{[2]} = 0.8 \\ \beta_2^{[2]} = 0.18 \end{cases}$$

5) By applying Remark 1, one can obtain

$$\delta_1^{[1]} = \delta_2^{[1]} = 1.25,\ \delta_1^{[2]} = \delta_2^{[2]} = 1.11$$

Then by Theorem 1, $\varsigma^{[1]} = -0.36$ and $\varsigma^{[2]} = -0.29$, and the almost sure consensus can be reached if $T_{\min}^{[1]} \geq -\mu^{[1]}/\varsigma^{[1]} = 13.4$ and $T_{\min}^{[2]} \geq -\mu^{[2]}/\varsigma^{[2]} =$

15.9.

To verify the correctness of the proposed result, construct the following deterministic switching signal $\gamma(k)$, as shown in Figure.3,

$$\gamma(k) = \begin{cases} 2, & k \in [mT, mT+16) \\ 1, & k \in [mT+16, mT+30) \end{cases}, m = 0, 1, \ldots$$

where $T = 30$.

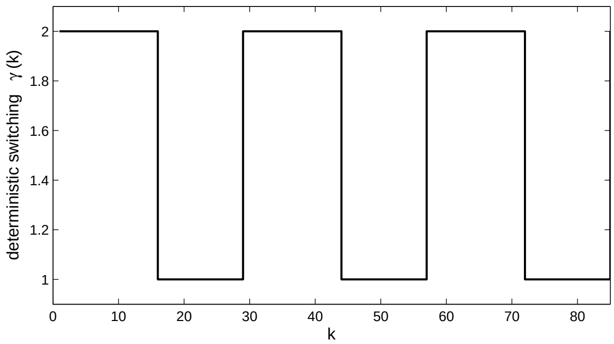

Figure 3: Deterministic switching $\gamma(k)$

Four realizations of the Markovain switching $\{\sigma(k)\}$ and the relative difference of the states of agents are presented in Figure 4 and Figure 5 respectively, which demonstrates the almost sure consensus is reached for this MAS.

6. Conclusion

This paper deals with almost sure consensus for the discrete-time MAS with the structure of two level switching. The deterministic change of the agent dynamics makes the lower level switching, while the upper level switching represents the stochastic jumps of connection topologies. Moreover, the transition probability of the stochastic topology jumps are influenced by the deterministic switchings of the low level. Based on the analysis of the transient properties of discrete-time Markov chain and by applying the method of dwell time, a sufficient condition of almost sure consensus for this MAS is proposed. A numerical example is finally presented to demonstrate the effectiveness of the developed protocol design approach.

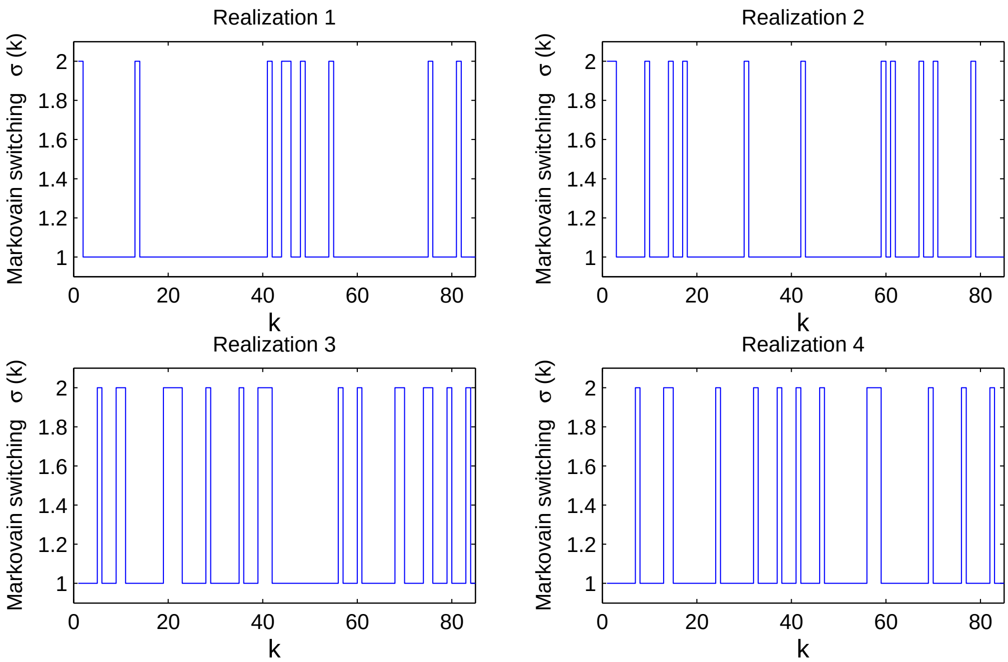

Figure 4: Four realizations of the Markovain switching $\{\sigma(k)\}$

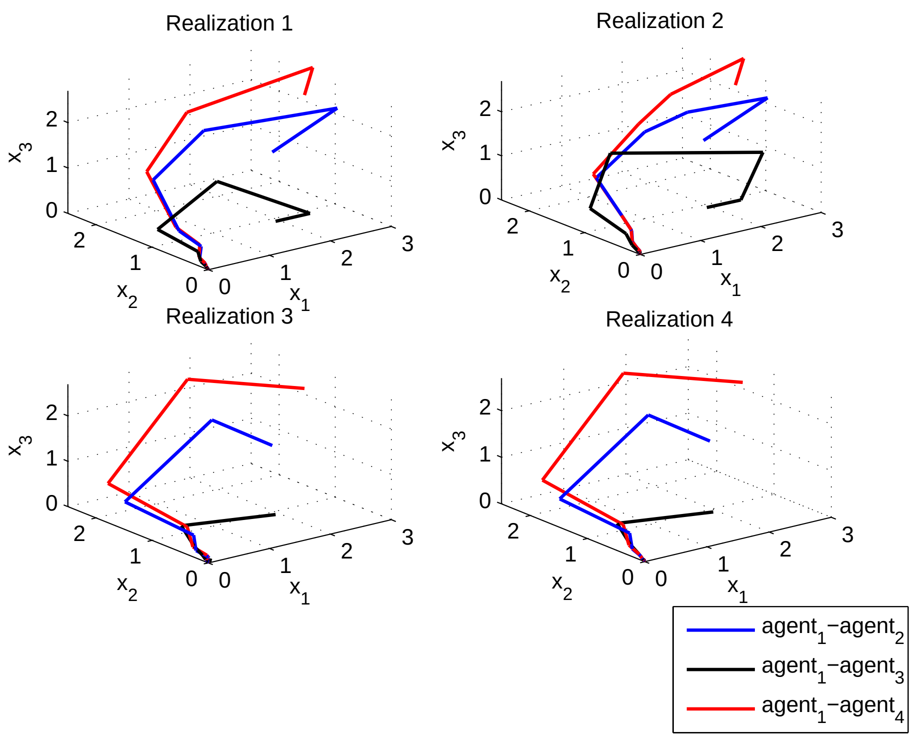

Figure 5: Four realizations of the relative difference of agents

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Appendix 1:

The parameters c and $K^{[\gamma(k)]}$ in protocol (3) can be calculated by the following three steps:

1) Solving the following Riccati inequality[25]to obtain $Q^{[j]}$:

$$A^{[j]^T}Q^{[j]}A^{[j]} - \chi_c^{[j]}A^{[j]^T}Q^{[j]}B^{[j]}(B^{[j]^T}Q^{[j]}B^{[j]})^{-1}B^{[j]^T}Q^{[j]}A^{[j]} - Q^{[j]} < 0 \quad (25)$$

where $\chi_c^{[j]} \in [0, 1)$ is constant.

2) Choose the coupling strength c from the set

$$c \in \left\{ c \in R^+ : 1 - |1 - c\lambda_k(L_i)|^2 > \max_{1 \le j \le s} \chi_c^{[j]} \right\} \quad (26)$$

where $1 \le i \le q, 1 \le k \le N, \lambda_k(L_i) \ne 0$.

3) Construct the matrix $K^{[j]}$ by

$$K^{[j]} = (B^{[j]^T}Q^{[j]}B^{[j]})^{-1}B^{[j]^T}Q^{[j]}A^{[j]} \quad (27)$$

By the following Lemma 3, it is clear that with the designed c and $K^{[j]}$, $F_i^{[j]}$ is Schur if $\mathcal{G}_i$ is connected.

Lemma 3: For $F = I_{N-1} \otimes A - cWLY \otimes BK$, the parameters are as in (7), the two statements as below are true:

1) F is Schur if and only if digraph $\mathcal{G}$ is connected and $\rho(A - c\lambda_i(L)BK) < 1$, $i = 2, \ldots, N$, where $\lambda_i(L)$ are the nonzero eigenvalues of Laplacian matrix L.

2) If $\mathcal{G}$ is connected and coupling strength c satisfies $1 - |1 - c\lambda_i(L)|^2 > \chi_c$, $i = 2, \ldots, N$, then $\rho(A - c\lambda_i(L)BK) < 1$ are fulfilled by choosing $K = (B^TQB)^{-1}B^TQA$, where Q satisfies the following Riccati inequality.

$$A^TQA - \chi_cA^TQB(B^TQB)^{-1}B^TQA - Q < 0$$

Proof. From the definition of T in Section 3 and noticing $L\mathbf{1}_N = 0$, one can obtain

$$TLT^{-1} = \left[\begin{array}{cc} 0 & l^1 Y \\ \mathbf{0}_{N-1} & WLY \end{array} \right]$$

where l^1 is the first row of L.

Then it can seen that matrix WLY has the same eigenvalues with Laplacian matrix L except the zero eigenvalue $\lambda_1(L) = 0$. Hence when $\mathcal{G}$ is connected, there exist a nonsingular matrix M such that $WLY = M\Lambda M^{-1}$, where Λ is an upper-triangular matrix, and the diagonal entries of Λ are same as the eigenvalues of L excluding zero eigenvalues $\lambda_1(L)$. Thus, F can be rewritten as

$$F = (M \otimes I_n)\left(I_{N-1} \otimes A - c\Lambda \otimes BK\right)\left(M^{-1} \otimes I_n\right)$$

Denote $\mathcal{F} := I_{N-1} \otimes A - c\Lambda \otimes BK$. Note that the elements of $\mathcal{F}$ are either block diagonal or block upper triangular. Hence, $\mathcal{F}$ is Schur if and only if the sub-matrix on the diagonal position of $\mathcal{F}$, i.e. $A - c\lambda_i(L)BK$ are Schur, $i = 2, \ldots, N$, which is further equivalent to that $\rho(A - c\lambda_i(L)BK) < 1$.

Moreover, let $K = (B^TQB)^{-1}B^TQA$ and denote $\chi_i = 1 - |1 - c\lambda_i(L)|^2$. It is found that

$$\begin{aligned} &(A - c\lambda_i(L)BK)^H Q (A - c\lambda_i(L)BK) - Q \\ &= A^TQA - \chi_i A^TQB(B^TQB)^{-1}B^TQA - Q \end{aligned} \tag{28}$$

where $[*]^H$ denotes the conjugate transpose of matrix $[*]$.

Define $x(k+1) = (A - c\lambda_i(L)BK)\, x(k)$ and $V(k) = x^T(k)Qx(k)$. By the condition $\chi_i > \chi_c$ and (28), one can obtain that $V(k+1) < V(k)$ for any $k \geq 0$. Then, the result $\rho(A - c\lambda_i(L)BK) < 1$ follows.$\square$

Appendix 2:

This appendix gives the proof of Lemma 2:

Following the similar derivation procedure in proposition 1 in [26], one can see that

$$E_{\boldsymbol{f}}\left[T_i\left(0,k\right)\right] = f_i + \pi_i(k-1) + \left(\boldsymbol{f}-\boldsymbol{\pi}\right)^T \sum_{q=0}^{k-2} P^q \boldsymbol{p}_i \tag{29}$$

$$\begin{aligned} E_{\boldsymbol{f}}\left[N_i\left(0,k\right)\right] &= (1-p_{ii})\left[f_i + \pi_i(k-1) + \left(\boldsymbol{f}-\boldsymbol{\pi}\right)^T \sum_{q=0}^{k-2} P^q \boldsymbol{p}_i\right] \\ &+p_{ii}\boldsymbol{f}^T P^{k-2}\boldsymbol{p}_i \end{aligned} \tag{30}$$

where $\boldsymbol{p}_i$ is the i-th column of matrix P.

Clearly, it holds the property that

$$\sum_{q=0}^{k-2} P^q \boldsymbol{p}_i = \left[\sum_{q=0}^{k-1} P^q\right]_i \tag{31}$$

where $\left[*\right]_i$ represents the i-th column of matrix $*$.

Notice that $(\sum_{q=0}^{k-1} P^q)(I-P) = I - P^k$ and $(\sum_{q=0}^{k-1} P^q)\mathbf{1}_{S\times S} = k\mathbf{1}_{S\times S}$, then

$$\left(\sum_{q=0}^{k-1} P^q\right)(I - P + \mathbf{1}_{S\times S}) = I - P^k + k\mathbf{1}_{S\times S}$$

That is

$$\sum_{q=0}^{k-1} P^q = (I-P^k)(I-P+\mathbf{1}_{S\times S})^{-1} + k\mathbf{1}_S\mathbf{1}_S^T(I-P+\mathbf{1}_{S\times S})^{-1} \tag{32}$$

From Proposition 5.7.1 in [27], one can get that the unique invariant distribution $\boldsymbol{\pi}^T = \mathbf{1}_S^T(I-P+\mathbf{1}_{S\times S})^{-1}$. Hence $\sum_{q=0}^{k-1} P^q$ can be rewritten as

$$\sum_{q=0}^{k-1} P^q = (I-P^k)(I-P+\mathbf{1}_{S\times S})^{-1} + k\mathbf{1}_S\boldsymbol{\pi}^T \tag{33}$$

Substituting (31) and (33) into (29), it follows

$$E_{\boldsymbol{f}}\left[T_i\left(0,k\right)\right] = f_i+\pi_i(k-1)+\left(\boldsymbol{f}-\boldsymbol{\pi}\right)^T\left[(I-P^k)(I-P+\mathbf{1}_{S\times S})^{-1}+k\mathbf{1}_S\boldsymbol{\pi}^T\right]_i$$

Note that $(\boldsymbol{f}-\boldsymbol{\pi})^T\left[k\mathbf{1}_S\boldsymbol{\pi}^T\right]_i = k\pi_i(\boldsymbol{f}-\boldsymbol{\pi})^T\mathbf{1}_S = 0$ and

$$\begin{aligned}
&(\boldsymbol{f}-\boldsymbol{\pi})^T\left[(I-P^k)(I-P+\mathbf{1}_{S\times S})^{-1}\right]_i \\
&\qquad = (\boldsymbol{f}-\boldsymbol{\pi})^T(I-P^k)\left[(I-P+\mathbf{1}_{S\times S})^{-1}\right]_i \\
&\qquad = \boldsymbol{f}^T(I-P^k)\left[(I-P+\mathbf{1}_{S\times S})^{-1}\right]_i
\end{aligned}$$

where the second equality holds since $\boldsymbol{\pi}^T P = \boldsymbol{\pi}^T$ and $\boldsymbol{\pi}^T(I-P^k) = 0$. Hence,

$$E_{\boldsymbol{f}}\left[T_i\left(0,k\right)\right] = f_i + \pi_i(k-1) + \boldsymbol{f}^T(I-P^k)\left[(I-P+\mathbf{1}_{S\times S})^{-1}\right]_i$$

Following a similar procedure, one can also obtain

$$\begin{aligned}
E_{\boldsymbol{f}}\left[N_i\left(0,k\right)\right] = {} & (1-p_{ii})\left(f_i + \pi_i(k-1) + \boldsymbol{f}^T(I-P^k)\left[(I-P+\mathbf{1}_{S\times S})^{-1}\right]_i\right) \\
& + p_{ii}\boldsymbol{f}^T\left[P^{k-1}\right]_i
\end{aligned}$$

This completes the proof.□

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